

HOMOGENEOUS ROTA-BAXTER OPERATORS ON A_ω (II)

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ABSTRACT. In this paper we study k -order homogeneous Rota-Baxter operators with weight 1 on the simple 3-Lie algebra A_ω (over a field of characteristic zero), which is realized by an associative commutative algebra A and a derivation Δ and an involution ω (Lemma 2.3). A k -order homogeneous Rota-Baxter operator on A_ω is a linear map R satisfying $R(L_m) = f(m+k)L_{m+k}$ for all generators $\{L_m \mid m \in \mathbb{Z}\}$ of A_ω and a map $f : \mathbb{Z} \rightarrow \mathbb{F}$, where $k \in \mathbb{Z}$. We prove that R is a k -order homogeneous Rota-Baxter operator on A_ω of weight 1 with $k \neq 0$ if and only if $R = 0$ (see Theorems 3.2), and R is a 0-order homogeneous Rota-Baxter operator on A_ω of weight 1 if and only if R is one of the forty possibilities which are described in Theorems 3.5, 3.7, 3.9, 3.10, 3.18, 3.21 and 3.22.

1. INTRODUCTION

Rota-Baxter operators have been closely related to many fields in mathematics and mathematical physics. They have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory [3, 4, 9, 10], as well as in the application of the renormalization method in solving divergent problems in number theory [16, 18], they are also important topics in many fields such as symplectic geometry, integrable systems, quantum groups and quantum field theory [1, 2, 8, 9, 12, 13, 14, 15, 17, 16, 19, 20].

Authors in [6] investigated the Rota-Baxter operators on n -Lie algebras [11] and studied the structure of Rota-Baxter 3-Lie algebras, and they also provided a method to realize Rota-Baxter 3-Lie algebras from Rota-Baxter 3-Lie algebras, Rota-Baxter Lie algebras, Rota-Baxter pre-Lie algebras and Rota-Baxter commutative associative algebras and derivations. In paper [5], authors discussed a class of Rota-Baxter operators of weight zero on an infinite dimensional simple 3-Lie algebra A_ω over a field \mathbb{F} of characteristic zero, which is the 0-order homogeneous Rota-Baxter operators of weight zero. A homogeneous Rota-Baxter operator on A_ω is a linear map R satisfying $R(L_m) = f(m)L_m$ for all generators $\{L_m \mid m \in \mathbb{Z}\}$ of A_ω and a map $f : \mathbb{Z} \rightarrow \mathbb{F}$. It is proved that R is a homogeneous Rota-Baxter operator on A_ω if and only if R is one of the five possibilities $R_{0_1}, R_{0_2}, R_{0_3}, R_{0_4}$ and R_{0_5} . By means of homogeneous Rota-Baxter operators, new 3-Lie algebras $(A, [, ,]_i)$ for $1 \leq i \leq 5$ are constructed, and R_{0_i} is also an homogeneous Rota-Baxter operator on the 3-Lie algebra $(A, [, ,]_i)$, for $1 \leq i \leq 5$, respectively.

In this paper we investigate k -order homogeneous Rota-Baxter operators of weight 1 on the simple 3-Lie \mathbb{F} -algebra A_ω , where \mathbb{F} is a field of characteristic zero. Throughout this paper, by an algebra we mean an \mathbb{F} -algebra and we denote by \mathbb{Z} the set of integers.

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2. PRELIMINARY

We recall that a **3-Lie algebra** over a field \mathbb{F} is an \mathbb{F} -vector space A endowed with a ternary multi-linear skew-symmetric operation satisfying for all $x_1, x_2, x_3, y_2, y_3 \in A$.

$$(1) \quad [[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3] + [[x_2, y_2, y_3], x_3, x_1] + [[x_3, y_2, y_3], x_1, x_2].$$

Definition 2.1. Let $\lambda \in \mathbb{F}$ be fixed. A **Rota-Baxter 3-algebra** is a 3-algebra (A, \langle, \rangle) with a linear map $R : A \rightarrow A$ such that

$$(2) \quad \begin{aligned} \langle R(x_1), R(x_2), R(x_3) \rangle &= R(\langle R(x_1), R(x_2), x_3 \rangle + \langle R(x_1), x_2, R(x_3) \rangle + \langle x_1, R(x_2), R(x_3) \rangle \\ &\quad + \lambda \langle R(x_1), x_2, x_3 \rangle + \lambda \langle x_1, R(x_2), x_3 \rangle + \lambda \langle x_1, x_2, R(x_3) \rangle \\ &\quad + \lambda^2 \langle x_1, x_2, x_3 \rangle). \end{aligned}$$

Lemma 2.2. Let (A, \langle, \rangle) be a 3-algebra over \mathbb{F} , $R : A \rightarrow A$ be a linear map and $\lambda \in \mathbb{F}$, $\lambda \neq 0$. Then (A, \langle, \rangle, R) is a Rota-Baxter 3-algebra of weight λ if and only if $(A, \langle, \rangle, \frac{1}{\lambda}R)$ is a Rota-Baxter 3-algebra of weight 1.

Proof. Apply Eq (2). □

Lemma 2.3. [7] Let A be an \mathbb{F} -vector space with a basis $\{L_n \mid n \in \mathbb{Z}\}$. Then A is a simple 3-Lie algebra in the multiplication

$$(3) \quad [L_l, L_m, L_n] = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix} L_{l+m+n-1}, \text{ for all } l, m, n \in \mathbb{Z}.$$

Notation. In the following, the 3-Lie algebra A in Lemma 2.3 is denoted by A_ω , and we set

$$(4) \quad D(l, m, n) := \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix}.$$

Lemma 2.4. [5] $D(l, m, n) = 0$ if and only if for all $l, m, n, k, s, t \in \mathbb{Z}$,

$$(l - m)(l - n)(m - n) = 0, \text{ or } l = 2k + 1, m = 2s + 1, n = 2t + 1, \text{ or } l = 2k, m = 2s, n = 2t.$$

3. HOMOGENEOUS ROTA-BAXTER OPERATORS OF WEIGHT 1 ON 3-LIE ALGEBRA A_ω

By Definition 2.1, if $(A, [,], R)$ is a Rota-Baxter 3-Lie algebra of weight 1, then the \mathbb{F} -linear map $R : A \rightarrow A$ satisfies, for all $x_1, x_2, x_3 \in A$,

$$(5) \quad \begin{aligned} [R(x_1), R(x_2), R(x_3)] &= R([R(x_1), R(x_2), x_3] + [R(x_1), x_2, R(x_3)] + [x_1, R(x_2), R(x_3)] \\ &\quad + [R(x_1), x_2, x_3] + [x_1, R(x_2), x_3] + [x_1, x_2, R(x_3)] \\ &\quad + [x_1, x_2, x_3]). \end{aligned}$$

Definition 3.1. Let R be a Rota-Baxter operator on the 3-Lie algebra A_ω . If there exist a map $f : \mathbb{Z} \rightarrow \mathbb{F}$, and $k \in \mathbb{Z}$ such that

$$(6) \quad R(L_m) = f(m + k)L_{m+k}, \quad \forall m \in \mathbb{Z},$$

then R is called a **k -order homogeneous Rota-Baxter operator**, which is denoted by R_k .

3.1. k -order homogeneous Rota-Baxter operators with $k \neq 0$. From Eq (6), we know that for all $x, y, z \in A_\omega$,

$$\begin{aligned}
& [R_k(L_l), R_k(L_m), R_k(L_n)] = [f(l+k)L_{l+k}, f(m+k)L_{m+k}, f(n+k)L_{n+k}] \\
& = f(l+k)f(m+k)f(n+k)D(l+k, m+k, n+k)L_{l+m+n+3k-1}, \\
& R_k([L_l, R_k(L_m), R_k(L_n)] + [R_k(L_l), L_m, R_k(L_n)] + [R_k(L_l), R_k(L_m), L_n] \\
& + [R_k(L_l), L_m, L_n] + [L_l, R_k(L_m), L_n] + [L_l, L_m, R_k(L_n)] + [L_l, L_m, L_n]) \\
& = R_k([L_l, f(m+k)L_{m+k}, f(n+k)L_{n+k}] + [f(l+k)L_{l+k}, L_m, f(n+k)L_{n+k}] \\
& + [f(l+k)L_{l+k}, f(m+k)L_{m+k}, L_n]) + [f(l+k)L_{l+k}, L_m, L_n] \\
& + [L_l, f(m+k)L_{m+k}, L_n] + [L_l, L_m, f(n+k)L_{n+k}] + [L_l, L_m, L_n] \\
& = f(m+k)f(n+k)f(l+m+n+3k-1)D(l, m+k, n+k)L_{l+m+n+3k-1} \\
& + f(l+k)f(n+k)f(l+m+n+3k-1)D(l+k, m, n+k)L_{l+m+n+3k-1} \\
& + f(l+k)f(m+k)f(l+m+n+3k-1)D(l+k, m+k, n)L_{l+m+n+3k-1} \\
& + f(l+k)f(l+m+n+2k-1)D(l+k, m, n)L_{l+m+n+2k-1} \\
& + f(m+k)f(l+m+n+2k-1)D(l, m+k, n)L_{l+m+n+2k-1} \\
& + f(n+k)f(l+m+n+2k-1)D(l, m, n+k)L_{l+m+n+2k-1} \\
& + f(l+m+n+k-1)D(l, m, n)L_{l+m+n+k-1}.
\end{aligned}$$

Thanks to Eq (5),

$$\begin{aligned}
& [f(l+k)L_{l+k}, f(m+k)L_{m+k}, f(n+k)L_{n+k}] \\
& = R_k([L_l, f(m+k)L_{m+k}, f(n+k)L_{n+k}] + [f(l+k)L_{l+k}, L_m, f(n+k)L_{n+k}] \\
& + [f(l+k)L_{l+k}, f(m+k)L_{m+k}, L_n]).
\end{aligned}$$

Therefore, if $k \neq 0$, then for all $l, m, n \in \mathbb{Z}$, $R_k([L_l, L_m, L_n]) = 0$. Thanks to $A_\omega = [A_\omega, A_\omega, A_\omega]$, $R_k(A_\omega) = 0$.

This shows the following result.

Theorem 3.2. *A linear map R_k defined by Eq (6) is a k -order homogeneous Rota-Baxter operator of weight 1 on A_ω if and only if $R_k = 0$.*

3.2. 0-order homogeneous Rota-Baxter operators of weight 1. In the following we discuss the 0-order homogeneous Rota-Baxter operators of weight 1 on A_ω . Then Eq (6) is reduced to

$$(7) \quad R(L_m) = f(m)L_m, \forall m \in \mathbb{Z}.$$

For convenience, throughout this paper we suppose that R is a linear map on A_ω defined by Eq (7), and 0-order homogeneous Rota-Baxter operator R_0 of weight 1 on A_ω is simply denoted by R , and is simply called a **homogeneous Rota-Baxter operator on A_ω** .

Denote

$$\begin{aligned}
W_1 &= \{2m \mid m \in \mathbb{Z}, m \neq 0, f(2m) \neq 0\}, & U_1 &= \{2m+1 \mid m \in \mathbb{Z}, m \neq 0, f(2m+1) \neq 0\}, \\
W_2 &= \{2m \mid m \in \mathbb{Z}, m \neq 0, f(2m) = 0\}, & U_2 &= \{2m+1 \mid m \in \mathbb{Z}, m \neq 0, f(2m+1) = 0\}.
\end{aligned}$$

Lemma 3.3. *The linear map R is a homogeneous Rota-Baxter operator on A_ω if and only if the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that for all $l, m, n \in \mathbb{Z}$,*

$$\begin{aligned}
(8) \quad & f(2l+1)f(2m+1)f(2n) = (f(2l+1)f(2m+1) + f(2l+1)f(2n) \\
& + f(2m+1)f(2n) + f(2l+1) + f(2m+1) + f(2n) + 1)f(2l+2m+2n+1), l \neq m.
\end{aligned}$$

$$(9) \quad f(2l+1)f(2m)f(2n) = (f(2l+1)f(2m) + f(2l+1)f(2n) + f(2m)f(2n) \\ + f(2l+1) + f(2m) + f(2n) + 1)f(2l+2m+2n), m \neq n.$$

Proof. By Eq (5) and Eq (7), R is a homogeneous Rota-Baxter operator on A_ω if and only if f satisfies that for all $l, m, n \in \mathbb{Z}$,

$$f(l)f(m)f(n)D(l, m, n) = (f(l)f(m) + f(l)f(n) + f(m)f(n) + f(l) + f(m) \\ + f(n) + 1)f(l+m+n-1)D(l, m, n).$$

Follows from Lemma 2.4, we obtain the result. \square

From Eq (8) and Eq (9), for $l = n = 0$, and $m \in \mathbb{Z}, m \neq 0, 1$, we have

$$f(0)f(m)f(1) = (f(0)f(1) + f(m)f(1) + f(0)f(m) + f(0) + f(1) + f(m) + 1)f(m),$$

so we get

$$(10) \quad (f(0) + f(1) + 1)f(m)(f(m) + 1) = 0.$$

Therefore, we will start the discussion according to the value $f(0) + f(1) + 1$.

3.2.1. Homogeneous Rota-Baxter operators with $f(0) + f(1) + 1 \neq 0$. In this section we discuss homogeneous Rota-Baxter operators R on A_ω defined by Eq (7) of the case $f(0) + f(1) + 1 \neq 0$.

Lemma 3.4. *Let R be a homogeneous Rota-Baxter operator on A_ω . Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies equation*

$$(11) \quad f(m)(f(m) + 1) = 0, \forall m \in \mathbb{Z}, m \neq 0, 1.$$

Proof. The result follows from $f(0) + f(1) + 1 \neq 0$, and Eq (10), directly. \square

Theorem 3.5. *If at least one of the subsets $W_i, U_i, i = 1, 2$ is finite. Then R is a homogeneous Rota-Baxter operator on A_ω if and only if the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies one of the following, for all $m, n \in \mathbb{Z}$,*

- 1) $f(m) = 0$;
- 2) $f(m) = -1$;
- 3) $f(2m) = 0, f(2m+1) = -1, m \neq 0$, and $f(0)(f(1) + 1) = 0$;
- 4) $f(2m) = -1, f(2m+1) = 0, m \neq 0$ and $f(1)(f(0) + 1) = 0$.

Proof. If f satisfies one of the cases 1) - 4). By a direct computation, we know that R satisfies Eq (8) and Eq (9), that is, R is a homogeneous Rota-Baxter operator on A_ω .

Conversely, suppose that R is a homogeneous Rota-Baxter operator on A_ω .

First, we prove that if W_i (or U_i) is a finite subset, then W_i (or U_i) is empty, where $i = 1$ or 2 .

Without loss of generality, we can suppose that $|W_1| < \infty$.

If $|W_1| = s$, and $1 \leq s < \infty$. Suppose $W_1 = \{2m_0, \dots, 2m_{s-1}\}$, $s \geq 1$. Then $|W_2| = \infty$. Without loss of generality, we can suppose that $|U_1| \neq 0$. Then there is $n_0 \neq 0$ such that $f(2n_0 + 1) = -1$. We assert that $|U_2| < \infty$ and $|U_1| = \infty$.

In fact, if $|U_2| = \infty$. Then there exist $2m, 2n \in W_2$, and $2l + 1 \in U_2$ such that $m \neq n$ and $2m + 2n + 2l = 2m_0$. By Eq (9), we get the contradiction $0 = f(2m)f(2n)f(2l + 1) = f(2m_0)$.

Therefore, $|U_2| < \infty$, and $|U_1| = \infty$. So there exist $2l+1, 2n+1 \in U_1$, and $2m \in W_2$ such that $l \neq n$, and $2m+2n+2l = 2n_0$. We get the contradiction $0 = f(2m)f(2n+1)f(2l+1) = f(2n_0+1)$.

Summarizing above discussion, we obtain that W_1 is empty, that is, $f(2m) = 0$ for all $m \in \mathbb{Z}, m \neq 0$.

Second we discuss the characteristic of f .

• If U_2 is non-empty, then there is $2n_0+1 \in U_2$ such that $f(2n_0+1) = 0$. By Eq (8) and Eq (9), for all $m \neq -n_0$ and $m \neq 0$, f satisfies that

$$f(2n_0+1)f(2m)f(-2n_0-2m) = f(0) = 0,$$

$$f(2n_0+1)f(1)f(-2n_0) = (f(1)+1)f(1) = 0.$$

Thanks to $f(0) + f(1) + 1 \neq 0$, $f(0) = f(1) = 0$. Again by Eq (8), for all $m \in \mathbb{Z}$,

$$f(2n_0+1)f(1)f(2m) = f(2n_0+2m+1) = 0,$$

we obtain that for all $l \in \mathbb{Z}, l \neq -n_0$, $f(2l+1) = 0$. By the similar discussion to the above, we obtain that for all $l \in \mathbb{Z}, f(2l+1) = 0$. This is the case 1).

•• If U_2 is empty, then for all $l \in \mathbb{Z}, l \neq 0$, $f(2l+1) = -1$. Thanks to Eq (8) and Eq (9), $f(0)(f(1)+1) = 0$. This is the case 3).

••• Similarly, if W_2 is empty, then for all $m \in \mathbb{Z}, m \neq 0$, $f(2m) = -1$. By the similar discussion, we obtain the cases 2) and 4).

If U_1 is empty, then for all $m \in \mathbb{Z}, m \neq 0$, $f(2m+1) = 0$. We obtain the cases 1) and 4).

If U_2 is empty, then for all $m \in \mathbb{Z}, m \neq 0$, $f(2m+1) = -1$. We obtain the cases 2) and 3). \square

Now we discuss the case $|W_i| = |U_i| = \infty$, for $i = 1, 2$.

Lemma 3.6. *Let R be a homogeneous Rota-Baxter operator on A_ω . If $W_1 = \{2m_i | m_i < m_{i+1}, i \in \mathbb{Z}, i \geq 0\}$. Then $U_1 = \{2l_i + 1 | l_i < l_{i+1}, i \in \mathbb{Z}, i \geq 0\}$, and $l_0 \geq -m_1, l_1 \geq -m_0$.*

Proof. For all $2l+1 \in U_1$, by Eq (9), we have $f(2m_0+2m_1+2l) = -1$. Then $2l+2m_0+2m_1 \geq m_0$, we obtain $l \geq -m_1$. So we can suppose that $U_1 = \{2l_i + 1 | l_i < l_{i+1}, i \in \mathbb{Z}, i \geq 0\}$, where $l_0 \geq -m_1$. Similarly, by Eq (8), we get $m_0 \geq -l_1$. \square

From Lemma 3.6, Eq (8) and Eq (9), we need to discuss the following four cases:

(1) $l_0 = -m_1$.

By a direct computation according Eq (8) and Eq (9), we have

$$m_i = m_1 + (i-1)(m_1 - m_0), \quad l_i = -m_0 + (i-1)(m_1 - m_0), \quad i \in \mathbb{Z}, i \geq 1,$$

where $W_1 = \{2m_i | m_i < m_{i+1}, i = 0, 1, 2, \dots\}$, and $U_1 = \{2l_i + 1 | l_i < l_{i+1}, i = 0, 1, 2, \dots\}$.

(2) $-m_1 < l_0 < -m_0$.

From $2(m_0 + l_0 + m_1) \in W_1$, and $m_0 + l_0 + m_1 < m_1$, we have $m_0 + l_0 + m_1 = m_0$, this contradicts $l_0 < -m_1$. Therefore, this case does not exist.

(3) $l_0 = -m_0$.

From $f(0) = f(0)f(2m_0)f(2l_0+1) = f(0)f(2m_0)f(-2m_0+1) = -f(0)^2$,

$f(1) = f(1)f(2m_0)f(2l_0+1) = f(1)f(2m_0)f(-2m_0+1) = -f(1)^2$, and

$f(0) + f(1) + 1 \neq 0$, we have $f(0) = f(1) = 0$ or $f(0) = f(1) = -1$.

- If $f(0) = f(1) = 0$. Then for all $k, l \in \mathbb{Z}$, $k > 0$ and $l > 0$,

$$f(2m_0 - 2k)f(-2m_0 - 2l + 1)f(0) = f(-2(k + l)) = 0,$$

$$f(2m_0 - 2k)f(-2m_0 - 2l + 1)f(1) = f(-2(k + l) + 1) = 0,$$

we obtain $m_0 \geq 1$, $-m_0 = l_0 \geq -1$. We assert that

$$m_0 = 1, l_0 = -1.$$

In fact, if there is $k_0 > 1$ such that $f(2k_0) = 0$, then $f(-2k_0 - 2 + 1) = 0$. Thanks to Eq (8), we get the contradiction $f(1)f(2k_0)f(-2k_0 - 2 + 1) = f(-2 + 1) = f(2l_0 + 1) = 0$. Therefore,

$$W_1 = \{2k, k \in \mathbb{Z}, k > 0\}, \quad U_1 = \{-1, 2k + 1, k \in \mathbb{Z}, k > 0\}.$$

•• If $f(0) = f(1) = -1$. For all $l, m, n, s \in \mathbb{Z}$, $lmns \neq 0$, if $f(2l + 1) = f(2n + 1) = f(2m) = f(2s) = -1$, then $f(2l + 2n + 1) = f(2m + 2s) = f(2l + 2m) = f(2l + 2m + 1) = -1$. We obtain that $2m_1 + 2l_0 = 2m_1 - 2m_0 \in W_1$, $2l_1 + 2l_0 + 1 = 2l_1 - 2m_0 + 1 \in U_1$.

If $m_0 > 0$, by Lemma 3.6, $m_1 - m_0 > 0$, $l_1 - m_0 < l_1$. Then $m_1 = 2m_0$, $l_1 = m_0$. Inductively, suppose $m_k = (k + 1)m_0$, $l_k = km_0$. Since

$$m_{k-1} = km_0 = m_k - m_0 < m_{k+1} - m_0 < m_{k+1},$$

$$m_{k+1} = (k + 2)m_0, \quad l_{k-1} = (k - 1)m_0 = l_k - m_0 < l_{k+1} - m_0 < l_{k+1}.$$

Then $l_{k+1} = (k + 1)m_0$. Therefore,

$$W_1 = \{2km_0 \mid k \in \mathbb{Z}, k > 0\}, \quad U_1 = \{-2m_0 + 1, 2km_0 + 1 \mid k \in \mathbb{Z}, k > 0\}.$$

Similarly, if $m_0 < 0$, we have

$$W_1 = \{2m_0, -2km_0 \mid k \in \mathbb{Z}, k > 0\}, \quad U_1 = \{2km_0 + 1 \mid k \in \mathbb{Z}, k > 0\}.$$

(4) $l_0 > -m_0$.

We can choose $W_1 = \{2m_k \mid m_k < m_{k+1}, m_k \in \mathbb{Z}, k \geq 0\}$, and $U_1 = \{2l_k + 1 \mid l_k < l_{k+1}, k \geq 0\}$.

If there is $m' > m_0$ such that $f(2m') = 0$. From $m > m_0$, $-m' < -m_0 < l_0$, we have $f(-2m' + 1) = 0$. By Eq (8) and Eq (9),

$$f(0)f(2m')f(-2m' + 1) = (f(0) + 1)f(0) = 0,$$

$$f(1)f(2m')f(-2m' + 1) = (f(1) + 1)f(1) = 0.$$

Thanks to $f(0) + f(1) + 1 \neq 0$, $f(0) = f(1) = 0$, or $f(0) = f(1) = -1$.

- If $f(0) = f(1) = -1$. From $f(2m_0 + 2l_0) = -1$ and $f(2m_0 + 2l_0 + 1) = -1$, we obtain $m_0 > 0$, $l_0 > 0$.

In the case $m_0 = l_0$, from $f(2m_0) = -1$, we have $l_0 = m_0 > 1$. Therefore,

$$\{2km_0 \mid k \in \mathbb{Z}, k > 0\} \subseteq W_1, \text{ and } \{2km_0 + 1 \mid k \in \mathbb{Z}, k > 0\} \subseteq U_1.$$

If there is $0 < r < m_0$, $k > 0$ such that $f(2m_0k + 2r) = 0$. From $f(-2r) = f(-2km_0 + 1) = 0$, and Eq (9), we get the contradiction

$$0 = f(2m_0k + 2r)f(-2r)f(-2km_0 + 1) = f(0) = -1.$$

Therefore, $f(2m) \neq 0$, for all $m \geq m_0$, that is,

$$\{2km_0 \mid k \in \mathbb{Z}, k > 0\} = W_1.$$

Similarly we have $f(2m + 1) \neq 0$ for all $m \geq l_0$, that is,

$$\{2km_0 + 1 \mid k \in \mathbb{Z}, k > 0\} \subseteq U_1.$$

Thanks to Eq (8) and Eq (9), $f(2m) = f(2m+1) = -1$, $\forall m \in \mathbb{Z}, m \geq m_0$.

If $l_0 \neq m_0$. From $f(2l_0 + 2m_0) = f(2m_0 + 2l_0 + 1) = -1$, we have

$\{2km_0 + 2ln_0 \mid k, l \in \mathbb{Z}, k > 0, l \geq 0\} \subseteq W_1$, and $\{2km_0 + 2ln_0 + 1 \mid k, l \in \mathbb{Z}, k \geq 0, l > 0\} \subseteq U_1$.

By the similar discussion to the above, $W_1 = \{2m \mid m \in \mathbb{Z}, m \geq m_0\}$, and $U_1 = \{2n + 1 \mid n \in \mathbb{Z}, n \geq l_0\}$, and for all $l \in W_1 \cup U_1$, $f(l) = -1$.

•• Now we prove that the case $f(0) = f(1) = 0$ does not exist.

If f satisfies $f(0) = f(1) = 0$. From $l_0 > -m_0 > -m'$, $l_0 > -m' + 1$, we have $f(2m') = 0$, and

$$f(0)f(2m')f(-2m' + 2 + 1) = (f(0) + 1)f(2) = 0,$$

$$f(1)f(2m')f(-2m' + 2 + 1) = (f(0) + 1)f(3) = 0.$$

Then $f(2) = f(3) = 0$. For $k \in \mathbb{Z}, k > 0$, if $f(2k) = f(2k+1) = 0$, by Eq (8) and Eq (9), we have

$$f(0)f(2k)f(2+1) = (f(0) + 1)f(2k+2) = f(2k+2) = 0,$$

$$f(1)f(2k)f(2+1) = (f(0) + 1)f(2k+2+1) = f(2k+2+1) = 0.$$

Therefore, for all positive $k \in \mathbb{Z}$, $f(2k) = f(2k+2+1) = 0$, this contradicts $|U_1| = \infty$.

Summarizing above discussion, we obtain the following result.

Theorem 3.7. *Let R be a homogeneous Rota-Baxter operator on A_ω with $f(0) + f(1) + 1 \neq 0$, and $W_1 = \{2m_i \mid i \in \mathbb{Z}, i \geq 0, m_i < m_{i+1}\}$, $U_1 = \{2l_i + 1 \mid i \in \mathbb{Z}, i \geq 0, l_i < l_{i+1}\}$. Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) is one of the following cases:*

1) There exist $m_0, m_1 \in \mathbb{Z}$, $m_0 < m_1$ such that for all $k \in \mathbb{Z}, k \geq 0$,

$$f(2m_0) = f(2m_1 + 2k(m_1 - m_0)) = -1,$$

$$f(-2m_1 + 1) = f(-2m_0 + 2k(m_1 - m_0) + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

2) For all $k \in \mathbb{Z}, k > 0$,

$$f(2k) = f(-1) = f(2k+1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

3) There is $m_0 \in \mathbb{Z}$, $m_0 > 0$ such that for all $k \in \mathbb{Z}, k \geq 0$,

$$f(2km_0) = f(-2m_0 + 1) = f(2km_0 + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

4) There is $m_0 \in \mathbb{Z}$, $m_0 < 0$ such that for all $k \in \mathbb{Z}, k \leq 0$,

$$f(2m_0) = f(2km_0) = f(2km_0 + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

5) There exist $m_0, l_0 \in \mathbb{Z}$, $l_0 > -m_0$ such that for all $m, l \in \mathbb{Z}, m \geq m_0, l \geq l_0$,

$$f(2m) = f(2l + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

6) $f(0) = f(1) = -1$, and there is $m_0 \in \mathbb{Z}, m_0 > 1$ such that $f(m) = -1$, for all $m \in \mathbb{Z}, m \geq m_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

7) $f(0) = f(1) = -1$ and there exist $m_0, l_0 \in \mathbb{Z}$ such that $m_0 > 0, l_0 > 0, m_0 \neq l_0$, $f(2m) = f(2n + 1) = -1$, for all $m, n \in \mathbb{Z}, m \geq m_0, n \geq l_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

By the similar discussion, we get the following result.

Lemma 3.8. *Let R be a homogeneous Rota-Baxter operator on A_ω and $W_1 = \{2m_i \mid m_i > m_{i+1}, i \in \mathbb{Z}, i \geq 0\}$. Then $U_1 = \{2l_i + 1 \mid l_i > l_{i+1}, i \in \mathbb{Z}, i \geq 0\}$, and $l_0 \leq -m_1, l_1 \leq -m_0$.*

Proof. For all $2l + 1 \in U_1$, by Eq (9), $f(2m_0 + 2m_1 + 2l) = -1$. Then $2l + 2m_0 + 2m_1 \leq 2m_0$, and $l \leq -m_1$. So we can suppose $U_1 = \{2l_i + 1 \mid l_i > l_{i+1}, i \in \mathbb{Z}, i \geq 0\}$, $l_0 \leq -m_1$. Thanks to Eq (8), $m_0 \leq -l_1$. \square

Theorem 3.9. *Let R be a homogeneous Rota-Baxter operator on A_ω and*

$$W_1 = \{2m_i \mid i \in \mathbb{Z}, i \geq 0, m_i > m_{i+1}\}, U_1 = \{2l_i + 1 \mid i \in \mathbb{Z}, i \geq 0, l_i > l_{i+1}\}.$$

Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) is one of the following cases:

1) *There is $m_0, m_1 \in \mathbb{Z}$, $m_0 > m_1$ such that for all $k \in \mathbb{Z}, k \leq 0$,*

$$f(2m_0) = f(2m_1 + 2k(m_0 - m_1)) = -1,$$

$$f(-2m_1 + 1) = f(-2m_0 + 2k(m_0 - m_1) + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

2) *For all $k \in \mathbb{Z}, k < 0$,*

$$f(2) = f(2k) = f(2k + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

3) *There is $m_0 \in \mathbb{Z}$, $m_0 < 0$ such that for all $k \in \mathbb{Z}, k \geq 0$,*

$$f(2km_0) = f(-2m_0 + 1) = f(2km_0 + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

4) *There is $m_0 \in \mathbb{Z}$, $m_0 > 0$ such that for all $k \in \mathbb{Z}, k \leq 0$,*

$$f(2m_0) = f(2km_0) = f(2km_0 + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

5) *There exist $m_0, l_0 \in \mathbb{Z}$, $l_0 < -m_0$ such that for all $m, l \in \mathbb{Z}, m \leq m_0, l \leq l_0$,*

$$f(2m) = f(2l + 1) = -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

6) *$f(0) = f(1) = -1$, and there is $m_0 \in \mathbb{Z}, m_0 < -1$ such that $f(l) = -1$ for all $l \leq 2m_0 + 1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

7) *$f(0) = f(1) = -1$ and there exist $m_0, l_0 \in \mathbb{Z}$, $l_0 < 0$, $m_0 < 0$, $m_0 \neq l_0$ such that for all $m, l \in \mathbb{Z}, m \leq m_0, l \leq l_0$, $f(2m) = f(2l + 1) = -1$ and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

Proof. Apply the arguments used in the proof of Theorem 3.7. \square

Theorem 3.10. *If $\inf W_i = \inf U_i = -\infty$ and $\sup W_i = \sup U_i = +\infty$. Then R is a homogeneous Rota-Baxter operator on A_ω if and only if the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) is one of the following cases:*

1) *There is $m_0 \in \mathbb{Z}$, $m_0 \neq 0$ such that for all $k \in \mathbb{Z}$, $f(2km_0) = f(2m_0k + 1) = 0$, and $f(m) = -1$ for the remaining $m \in \mathbb{Z}$.*

2) *There is $m_0 \in \mathbb{Z}$, $m_0 \neq 0$ such that for all $k \in \mathbb{Z}$, $f(2km_0) = f(2m_0k + 1) = -1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

Proof. Let R be a homogeneous Rota-Baxter operator on A_ω . Suppose

$$W_2 = \{2m_i, 2m'_i | i \in \mathbb{Z}, i \geq 0\}, \quad U_2 = \{2l + 1, 2l'_i + 1 | i \in \mathbb{Z}, i \geq 0\},$$

where

$$\begin{aligned} \cdots < m'_{i+1} < m'_i < \cdots < m'_1 < m'_0 < 0 < m_0 < 2m_1 < \cdots < m_i < m_{i+1} < \cdots, \\ \cdots < l'_{i+1} < l'_i < \cdots < l'_1 < l'_0 < 0 < l_0 < l_1 < \cdots < l_i < l_{i+1} < \cdots. \end{aligned}$$

If $f(0) = b \neq 0, -1$. Thanks to Eq (8), for $l, k \in \mathbb{Z}$ and $l \neq k$, if $f(2l+1) = f(2k+1) = 0$, then $f(2l+2k+1) = 0$, and $f(2l_0+2l'_0+1) = 0$. Since $2l'_0+1 < 2l_0+2l'_0+1 < 2l_0+1$, $f(1) = 0$. By Eq (9), and $f(2m) = f(2n) = 0, m \neq n$, we have $f(2m+2n) = 0$. From $f(2m_0+2m'_0) = 0$, and $2m'_0 < 2m_0+2m'_0 < 2m_0$, we get the contradiction $f(0) = b = 0$. Therefore, $f(0) = 0$, or $f(0) = -1$.

If $f(0) = 0$. By $2l'_0+1 < 2l_0+2l'_0+1 < 2l_0+1$ and Eq (8), $f(0)f(2l_0+1)f(2l'_0+1) = f(2l_0+2l'_0+1) = 0$. Therefore, $l'_0 = -l_0$ and $f(1) = 0$.

Similar discussion, we obtain that for all $i \in \mathbb{Z}, i \geq 0$, $m_i = -m'_i$, $l_i = -l'_i$. Therefore, for all $2m, 2n \in W_2, 2l+1, 2s+1 \in U_2$, we have $2m+2n, 2m+2l \in W_2$ and $2l+2s+1, 2l+2m+1 \in U_2$. Since $0 < 2m_1-2m_0 = 2m_1+2m'_0 < 2m_1$, $2m_1-2m_0 = 2m_0$, and $m_1 = 2m_0$. Inductively, we have $m_i = (i+1)m_0$, $m'_i = -(i+1)m_0$, $l_i = (i+1)l_0$, $l'_i = -(i+1)l_0$, for all $i \in \mathbb{Z}, i \geq 0$.

We affirm $m_0 = l_0$.

In fact, if $m_0 \neq l_0$, then $m_0 - l_0 \neq 0$. From $2m_0 - 2l_0 = 2m_0 + 2l'_0 < 2m_0$, $2m_0 - 2l_0 \in W_2$, and $2l'_0+1 < 2m_0-2l_0+1 \in U_2$, we get the contradiction $2m_0-2l_0 < 0$ and $2m_0-2l_0 > 0$. Therefore, $m_0 = l_0$. We get case 1).

By the similar discussion, if $f(0) = -1$, then $f(1) = -1$, and we obtain the case 2). \square

3.2.2. Homogeneous Rota-Baxter operators with $f(0) = a \neq 0$ and $f(0) + f(1) + 1 = 0$. In this section we discuss homogeneous Rota-Baxter operators on A_ω of weight 1 defined by Eq (7) with $f(0) = a \neq 0$ and $f(0) + f(1) + 1 = 0$.

Lemma 3.11. *Let R be homogeneous Rota-Baxter operators on A_ω . Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that for all $l, m, n \in \mathbb{Z}$,*

- 1) $af(2l+1)f(2m+1) = ((a+1)f(2l+1) + (a+1)f(2m+1) + f(2l+1)f(2m+1) + (a+1))f(2l+2m+1), l \neq m.$
- 2) $-(a+1)f(2m+1)f(2n) = (-af(2m+1) - af(2n) + f(2m+1)f(2n) - a)f(2m+2n+1), m \neq 0.$
- 3) $af(2l+1)f(2m) = ((a+1)f(2l+1) + (a+1)f(2m) + f(2l+1)f(2m) + (a+1))f(2l+2m), m \neq 0.$
- 4) $-(a+1)f(2m)f(2n) = (-af(2m) - af(2n) + f(2m)f(2n) - a)f(2m+2n), m \neq n.$

Proof. The result follows from Eq (8) and Eq (9), directly. \square

Theorem 3.12. *Let R be a homogeneous Rota-Baxter operators on A_ω , then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies equation, for all $m \in \mathbb{Z}$,*

$$(12) \quad f(1-m) + f(m) + 1 = 0.$$

Proof. By 2) and 3) in Lemma 3.11, for all $m, n \in \mathbb{Z}, m \neq 0, n \neq 0$,

$$\begin{aligned} & -f(2m+1)f(2n) \\ &= f(2m+2n+1)(-af(2m+1)-af(2n)+f(2m+1)f(2n)-a) \\ &+ f(2m+2n)((a+1)f(2m+1)+(a+1)f(2n)+f(2m+1)f(2n)+a+1). \end{aligned}$$

Then in the case $m = -n$, we obtain $f(2m+1) + f(-2m) + 1 = 0$. The result follows. \square

Theorem 3.13. *Let R be a homogeneous Rota-Baxter operators on A_ω , and $f(2k) \neq 0, f(2l) \neq 0, f(2m+1) \neq 0, f(2n+1) \neq 0$, for $k, l, m, n \in \mathbb{Z}$ and $klmn \neq 0$. Then we have*

- 1) $f(2k+2l) \neq 0$; 2) $f(2k+2m) \neq 0$; 3) $f(2k+2m+1) \neq 0$;
- 4) $f(2m+2n+1) \neq 0$; 5) $f(2m+2n+2k+1) \neq 0$; 6) $f(2m+2k+2l) \neq 0$;
- 7) $f(1-2k+2m) \neq 0, m \neq -k$; 8) $f(4k) \neq 0$; 9) $f(1-2k-2m)+1 \neq 0$;
- 10) $f(2k-2m)+1 \neq 0$; 11) $f(1-4k)+1 \neq 0$.

Proof. The result 1) follows from 4) in Lemma 3.11 of the case $m = k, n = l, k \neq l$.

The result 2) follows from 2) in Lemma 3.11 of the case $m = m, n = k, k \neq 0$.

The result 3) follows from 3) in Lemma 3.11 of the case $l = m, m = k, m \neq 0$.

The result 4) follows from 1) in Lemma 3.11 of the case $l = m, m = n, m \neq n$.

The result 5) and 6) follows from Eq (8) and Eq (9), directly.

The result 7) follows from 1) in Lemma 3.11 of the case $l = 0, 2m+1, -2k+1, m \neq -k$.

The result 8) follows from 3) in Lemma 3.11 of the case $l = k, m = k, k \neq 0$.

The result 9), 10) and 11) follow from 2), 7) and 10) and Eq (12), respectively. \square

Lemma 3.14. *If at least one of the subsets $W_i, U_i, i = 1, 2$ is finite. Then R is not a homogeneous Rota-Baxter operator on A_ω .*

Proof. The result follows from 1), 2), 3) and 4) in Theorem 3.13, directly. \square

Theorem 3.15. *If R is a homogeneous Rota-Baxter operator on A_ω , then*

$$\inf W_i = \inf U_i = -\infty, \sup W_i = \sup U_i = +\infty,$$

and there is $m_0 \in \mathbb{Z}, m_0 \neq 0$ such that

$$(13) \quad W_1 = \{2m_0k | k \in \mathbb{Z}, k \neq 0\}, U_1 = \{2m_0k + 1 | k \in \mathbb{Z}\}.$$

Proof. If there is $m_0 \in \mathbb{Z}$ such that $f(2m_0) \neq 0$, and for all $2m \in W_1, 2m \geq 2m_0$ (similar discussion for the case $2m \leq 2m_0$). By 2) and 8) in Theorem 3.13, and Lemma 3.14, for all $2m+1 \in U_1$, we have $f(2m+2m_0) \neq 0$, and $f(4m_0) \neq 0$. Then $4m_0 > 2m_0, 2m+2m_0 \geq 2m_0$. We obtain that $m_0 > 0$, and there is $l_0 \in \mathbb{Z}, l_0 > 0$ such that for all $2l+1 \in U_1, 2l+1 \geq 2l_0+1$. From 7) in Theorem 3.13, $f(1+2l_0-2m_0) \neq 0$. We get the contradiction $2l_0+1 \leq 1+2l_0-2m_0 < 1+2l_0$. Therefore, $\inf W_i = \inf U_i = -\infty, \sup W_i = \sup U_i = +\infty$.

Then we can suppose

$$W_1 = \{2m_i, 2m'_i | i \in \mathbb{Z}, i \geq 0\}, U_1 = \{2l+1, 2l'_i+1 | i \in \mathbb{Z}, i \geq 0\},$$

where

$$\begin{aligned} \cdots < m'_{i+1} < m'_i < \cdots < m'_1 < m'_0 < 0 < m_0 < m_1 < \cdots < m_i < m_{i+1} < \cdots, \\ \cdots < l'_{i+1} < l'_i < \cdots < l'_1 < l'_0 < 0 < l_0 < l_1 < \cdots < l_i < l_{i+1} < \cdots, \end{aligned}$$

Thanks to Theorem 3.13, $2m_0 + 2m'_0 \in W_1$, $m'_0 < m_0 + m'_0 < m_0$. We obtain $m'_0 = -m_0$. Since $0 < 2m_1 + m'_0 = 2m_1 - 2m_0 < 2m_1$, $m_1 = 2m_0$. Inductively, we get

$$m_i = (i+1)m_0, \quad m'_i = -(i+1)m_0, \quad i \in \mathbb{Z}, i \geq 0.$$

Similar discussion, for all $i \in \mathbb{Z}, i \geq 0$, $l_i = (i+1)l_0$ and $l'_i = -(i+1)l_0$.

From 2) and 3) in Theorem 3.13, there exist positive $s, t \in \mathbb{Z}$ such that

$$2l_0 + 2m_0 = 2sm_0 = 2tl_0.$$

Then $l_0 = (s-1)m_0, m_0 = (t-1)l_0$. It follows $l_0 = m_0$. The proof is complete. \square

The subset $T_{m_0} = W_1 \cup U_1$ is called the m_0 -**supporter of the homogeneous Rota-Baxter operator R** . Then for all $m \in \mathbb{Z}, m \neq 0, 1$, $f(m) \neq 0$ if and only if $m \in T_{m_0}$.

Corollary 3.16. *Let R be a homogeneous Rota-Baxter operator. Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that for $k \in \mathbb{Z}$, if $f(2m_0k) \neq 0$, then $f(2km_0) \neq -1$, $f(1+2km_0) \neq 0, -1$, and*

$$(14) \quad \frac{1}{f(2m_0k)} + \frac{1}{f(-2m_0k)} + \frac{1}{f(2m_0k)f(-2m_0k)} = \frac{1+2a}{a^2},$$

where $f(0) = a \neq 0$.

Proof. From 9) and 10) in Theorem 3.13, if $f(2m_0k) \neq 0$, then $f(2km_0) \neq -1$, $f(1+2km_0) \neq 0, -1$. Thanks to 4) in Lemma 3.11, for $m = -n = 2m_0k, k \in \mathbb{Z}, k \neq 0$, we have

$$-(1+a)f(2m_0k)f(-2m_0k) = -a^2f(2m_0k) - a^2f(-2m_0k) + af(2m_0k)f(-2m_0k) - a^2.$$

Since $\frac{1}{f(0)} = \frac{1}{a}$, we obtain Eq (14). \square

Corollary 3.17. *Let R be a homogeneous Rota-Baxter operator with m_0 -supporter T_{m_0} . Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that for all $k_1, k_2, k_3 \in \mathbb{Z}, k_2 \neq k_3$,*

$$(15) \quad \begin{aligned} & \frac{1}{f(2m_0k_1)} + \frac{1}{f(2m_0k_1)f(2m_0(-k_1+k_2+k_3))} + \frac{1}{f(2m_0(-k_1+k_2+k_3))} \\ &= \frac{1}{f(2m_0k_2)} + \frac{1}{f(2m_0k_3)f(2m_0k_2)} + \frac{1}{f(2m_0k_3)}. \end{aligned}$$

Proof. The result follows from 9) and 10) in Theorem 3.13, and Theorem 3.12. \square

Theorem 3.18. *Let R be a homogeneous Rota-Baxter operator on A_ω . Then there is $m_0 \in \mathbb{Z}, m_0 \neq 0$ such that the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies one of the two cases:*

1) For all $k \in \mathbb{Z}$, $f(2m_0k) = f(0) = a$, $f(2m_0k+1) = f(1) = -1-a$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

2) If there is $k_0 \in \mathbb{Z}, k_0 \neq 0$ such that $f(2m_0k_0) \neq a$, then $a \neq -1, -\frac{1}{2}$, and for all $k \in \mathbb{Z}$,

$$f(4m_0k) = a, \quad f(4m_0k+1) = -1-a,$$

$$(16) \quad f(4m_0k+2) = \frac{-a}{1+2a}, \quad f(4m_0k+3) = -\frac{1+a}{1+2a},$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

Proof. If for all $k \in \mathbb{Z}$, $f(2m_0k) = a$, then we get the case 1).

Now we prove the case 2).

By Theorem 3.15, if R is a homogeneous Rota-Baxter operator, then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that there is $m_0 \in \mathbb{Z}$, $m_0 \neq 0$ such that for all $l, k \in \mathbb{Z}$, $f(l) \neq 0$ if and only if $l = 2m_0k$ or $l = 2m_0k + 1$. From Theorem 3.15, and 9) and 10) in Theorem 3.13, for all $k \in \mathbb{Z}$,

$$f(2km_0) \neq -1, f(1 + 2km_0) \neq -1.$$

Then for $k \neq 0$, let $k_1 = k_3 = k$, $k_2 = -k_1$ in Eq (15), we obtain

$$(17) \quad f(2m_0k) = f(-2m_0k).$$

Thanks to Eq (12), for all $k \in \mathbb{Z}$, $k \neq 0$,

$$(18) \quad f(1 + 2m_0k) = f(1 - 2m_0k) = -1 - f(2m_0k).$$

From Eq (9), and Eq (12), for all nonzero $l, k \in \mathbb{Z}$, and $l \neq k$, we have

$$(19) \quad (f(2m_0k) - f(2m_0l))(f(2m_0k) + 2f(2m_0k)f(2m_0l) + f(2m_0l)) = 0,$$

$$(20) \quad (f(2m_0k) - a)(f(2m_0k) + 2af(2m_0k) + a) = 0.$$

Follows from Eq (19), Eq (20), Eq (14), if $f(2m_0l) \neq a$, then $a \neq -1, \frac{-1}{2}$, and

$$f(2m_0l) = f(-2m_0l) = \frac{-a}{1+2a}, f(2m_0l + 1) = f(-2m_0l + 1) = -\frac{1+a}{1+2a}.$$

If there exist $n_0, k_0 \in \mathbb{Z}$, $k_0 \neq 0$, $n_0 \neq 0$ such that $f(2m_0k_0) \neq a$ and $f(2m_0n_0) = a$, then $k_0 \neq n_0$ and $k_0 \neq -n_0$. Thanks to Eq (9), $f(2m_0(n_0 + k_0)) \neq a$.

Similar discussion, for $n_1, k_1 \in \mathbb{Z}$, $k_1 \neq k_0$, $n_1 \neq n_0$, if $f(2m_0n_1) = a$, $f(2m_0k_1) \neq a$, by Eq (15), $f(2m_0(k_0 + k_1)) = f(2m_0(n_0 + n_1)) = a$. Without loss of generality, we can suppose that $m_0 > 0$. And let $k_0, n_0 \in \mathbb{Z}$ be the least positive satisfying that $f(2m_0k_0) \neq a$ and $f(2m_0n_0) = a$. By the above discussion and Eq (17), $f(2m_0(k_0 - n_0)) \neq a$. Since $k_0 - n_0 < k_0$, $k_0 < n_0$, and $k_0 = 1$. If $n_0 > 2$, then $f(2m_02) \neq a$, and $f(2m_0(1 + 2)) = a$, we obtain $n_0 = 3$. From $f(2m_0(2 + 3)) \neq a$, and $f(2m_02) \neq a$, we have $f(2m_0(2 + 5)) = a$. From $f(2m_0(1 + 3)) \neq a$, $f(2m_03) = a$, we get the contradiction $f(2m_0(3 + 4)) \neq a$.

Therefore, we have $n_0 = 2$, and that $f(2m_0k) = a$ if and only if $k = 2l$, and $f(2m_0k) \neq a$ if and only if $k = 2l + 1$, where $l \in \mathbb{Z}$. Eq (16) follows. \square

3.2.3. Homogeneous Rota-Baxter operators with $f(0) = 0$ and $f(1) = -1$. In this section we discuss homogeneous Rota-Baxter operators R on A_ω with $R(L_0) = 0$, and $R(L_1) = -L_1$. Thanks to Eq (7), the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ satisfies $f(0) = 0$ and $f(1) = -1$.

Lemma 3.19. *Let R be a homogeneous Rota-Baxter operator on A_ω with $R(L_0) = 0$, and $R(L_1) = -L_1$. Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies following conditions, for all $l, m, n \in \mathbb{Z}$,*

$$1) (f(2l + 1) + 1)(f(2m + 1) + 1)f(2l + 2m + 1) = 0, l \neq m.$$

$$2) f(2m + 1)f(2n)(1 + f(2m + 2n + 1)) = 0, m \neq 0.$$

$$3) (f(2l + 1) + 1)(f(2m) + 1)f(2l + 2m) = 0, m \neq 0.$$

$$4) f(2m)f(2n)(1 + f(2m + 2n)) = 0, m \neq n.$$

Proof. The result follows from Eq (8), Eq (9), $f(0) = 0$ and $f(1) = -1$, directly. \square

Corollary 3.20. *Let R be a homogeneous Rota-Baxter operator on A_ω with $R(L_0) = 0$, and $R(L_1) = -L_1$. Then the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies that for all $k, l, m, n \in \mathbb{Z}$, $klmn \neq 0$,*

- 1) *if $f(2k) \neq 0, f(2l) \neq 0, k \neq l, k \neq -l$, then $f(2k + 2l) = -1$.*
- 2) *If $f(2k) \neq 0, f(2m + 1) \neq 0, m \neq 0$, then $f(2k + 2m + 1) = -1$.*
- 3) *If $f(2k) = 0, f(2n + 1) = 0, k \neq 0$, then $f(2k + 2n) = 0$.*
- 4) *If $f(2m + 1) = 0, f(2n + 1) = 0, m \neq n, m \neq -n$, then $f(2m + 2n + 1) = 0$.*
- 5) *If $k \neq 0, f(2k)f(-2k) = 0$.*
- 6) *For all $m \in \mathbb{Z}$, $(f(2m + 1) + 1)(f(-2m + 1) + 1) = 0$.*
- 7) $|W_2| = |U_1| = \infty$.

Proof. The result follows from Lemma 3.19, directly. \square

Theorem 3.21. *If $|W_1| < \infty$, then R is a homogeneous Rota-Baxter operator on A_ω if and only if the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) satisfies one of the following conditions:*

- 1) $|W_1| = |U_2| = 0$, and for all $m \in \mathbb{Z}$, $f(2m) = 0, f(2m + 1) = -1$.
- 2) $|W_1| = |U_2| = 0$, and there is nonzero $n_0 \in \mathbb{Z}$ such that $f(2n_0 + 1) \neq 0, -1$, and for all $m, n \in \mathbb{Z}$, $f(2m) = 0, f(2n + 1) = -1, n \neq n_0$.
- 3) $|W_1| = 0, |U_2| = 1$, and there is nonzero $n_0 \in \mathbb{Z}$ such that $f(2n_0 + 1) = 0$, and for all $m, n \in \mathbb{Z}$, $f(2m) = 0, f(2n + 1) = -1, n \neq n_0$.
- 4) $|W_1| = 1, |U_2| = 0$, and there is nonzero $m_0 \in \mathbb{Z}$ such that $f(2m_0) \neq 0$ and for all $m, n \in \mathbb{Z}$, $f(2m) = 0, f(2n + 1) = -1, m \neq m_0$.

Proof. The discussion is completely similar to Theorem 3.5. \square

From Theorem 3.21, Let R be a homogeneous Rota-Baxter operator with $R(L_0) = 0$ and $R(L_1) = -L_1$. Then $|W_1| \neq 0$ and $|U_2| \neq 0$ if and only if $|W_1| = |U_2| = \infty$. So in the following we discuss the case $|W_1| = |U_2| = \infty$.

Theorem 3.22. *Let $|W_1| = \infty$, then R is a homogeneous Rota-Baxter operator with $R(L_0) = 0$ and $R(L_1) = -L_1$ if and only if the map $f : \mathbb{Z} \rightarrow \mathbb{F}$ in Eq (7) is one of the following cases:*

(1) *There is $m_0, n_0 \in \mathbb{Z}$, $m_0 > 0, n_0 < 0$ such that for all $m, n \in \mathbb{Z}$, $f(2m) = 0$ if and only if $m < m_0$, and $f(2n + 1) = 0$ if and only if $n \leq n_0$. And f is one of the following seven cases:*

- 1) $f(2n + 1) = f(2m) = -1$, for all $n > n_0, m \geq m_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.
- 2) *There exist $c, d \in \mathbb{F}$, $cd \neq 0$ and $c \neq -1$, or $d \neq -1$ such that for all $m, n \in \mathbb{Z}, m \geq m_0$*

$$f(2m) = -1, f(2n + 1) = -1, f(-1) = c, f(-3) = d, n \geq 0,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$ (in this case $n_0 = -3$).

- 3) *There exist $c' \in \mathbb{F}$, $c' \neq 0$ and $c' \neq -1$, for all $m, n \in \mathbb{Z}, m \geq m_0, n \geq 0, n \neq 1$,*

$$f(2m) = -1, f(2n + 1) = f(-1) = f(-3) = -1, f(3) = c',$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$ (in this case $n_0 = -3$).

4) There is $g \in \mathbb{F}$, $g \neq 0, -1$ such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n \geq 0$,

$$f(2m) = -1, f(2n+1) = -1, f(-1) = g,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$ (in this case $n_0 = -2$).

5) There is $m_1 \in \mathbb{Z}$, $m_1 \geq m_0$, $h \in \mathbb{F}$, $h \neq 0, -1$ such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n > n_0$,

$$f(2m_1) = h, f(2m) = -1, f(2n+1) = -1, m \neq m_1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

6) There is $m_1, n_1 \in \mathbb{Z}$, $m_1 \geq m_0$, $n_1 > n_0$, $h, h' \in \mathbb{F}$, $h, h' \neq -1$ and $hh' \neq 0$ such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n > n_0$,

$$f(2m_1) = h, f(2n_1+1) = h', f(2m) = -1, f(2n+1) = -1, m \neq m_1, n \neq n_1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

7) There is $m_1, m_2 \in \mathbb{Z}$, $m_1, m_2 \geq m_0$, $m_1 \neq m_2$, and $g, r \in \mathbb{F}$, $g, r \neq -1$, $gr \neq 0$, such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n > n_0$,

$$f(2m_1) = g, f(2m_2) = r, f(2n+1) = f(2m) = -1, m \neq m_1, m_2,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

(2) There exist $m_0, n_0 \in \mathbb{Z}$, $m_0 < 0$ and $n_0 > 0$ such that for all $m, n \in \mathbb{Z}$, $f(2m) = 0$ if and only if $m > m_0$, and $f(2n+1) = 0$ if and only if $n \geq n_0$. And f is one of the seven cases:

1)' $f(2n+1) = -1$, and $f(2m) = -1$ for all $n < n_0$, $m \leq m_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

2)' There is $c \in \mathbb{F}$, $c \neq 0$ and $c \neq -1$ such that for all $m, n \in \mathbb{Z}$, $m \leq m_0$,

$$f(2m) = -1, f(2n+1) = -1, f(3) = c, n \leq 0,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

3)' There exist $c', d' \in \mathbb{F}$, $c'd' \neq 0$ and $c' \neq -1$, or $d' \neq -1$ such that for all $m, n \in \mathbb{Z}$, $m \leq m_0, n < -2$,

$$f(2m) = -1, f(2n+1) = f(1) = f(3) = -1, f(-1) = c', f(-3) = d',$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

4)' There is $g \in \mathbb{F}$, $g \neq 0, -1$ such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n \leq 0$,

$$f(2m) = -1, f(2n+1) = -1, f(-1) = g, n \neq -1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

5)' There exist $m_1 \in \mathbb{Z}$, $m_1 \leq m_0$, $h \in \mathbb{F}$, $h \neq 0, -1$ such that for all $m, n \in \mathbb{Z}$, $m \geq m_0, n < n_0$,

$$f(2m_1) = h, f(2m) = -1, f(2n+1) = -1, m \neq m_1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

6)' There exist $m_1, n_1 \in \mathbb{Z}$, $m_1 \leq m_0$, $n_1 < n_0$, $h, h' \in \mathbb{F}$, $h, h' \neq -1$ and $hh' \neq 0$ such that for all $m, n \in \mathbb{Z}$, $m \leq m_0, n < n_0$,

$$f(2m_1) = h, f(2n_1+1) = h', f(2m) = -1, f(2n+1) = -1, m \neq m_1, n \neq n_1,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

7)' There exist $m_1, m_2 \in \mathbb{Z}$, $m_1, m_2 \leq m_0$, $m_1 \neq m_2$, and $g, r \in \mathbb{F}$, $g, r \neq -1$, $gr \neq 0$ such that for all $m, n \in \mathbb{Z}$, $m \leq m_0$, $n < n_0$,

$$f(2m_1) = g, f(2m_2) = r, f(2n+1) = f(2m) = -1, m \neq m_1, m_2,$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

Proof. (i). We first discuss W_i and U_i , for $i = 1, 2$.

Since $|W_1| = \infty$, without loss of generality, we can suppose that there is $m \in \mathbb{Z}$, $f(2m) \neq 0$ and $m > 0$.

Then there is $2m_0 \in \mathbb{Z}$ such that $2m_0$ is the least positive which is contained in W_1 . We will prove that $W_1 = \{2m | m \in \mathbb{Z}, m \geq m_0\}$ and $U_2 = \{2n+1 | n \in \mathbb{Z}, n \leq n_0\}$.

If for all $n < 0$, $f(2n+1) \neq 0$. By Corollary 3.20, $f(2n+k2m_0+1) = -1$, for all $k \in \mathbb{Z}$, $k > 0$. We get the contradiction $|U_2| = 0$. Therefore, there is the largest negative $2n_0+1 \in \mathbb{Z}$ such that $f(2n_0+1) = 0$, that is, $2n_0+1 \in U_2$, $n_0 < 0$.

First, if there is $m \in \mathbb{Z}$, $m < 0$ such that $2m \in W_1$. Let $2m'_0 \in \mathbb{Z}$ be the largest negative which is contained in W_1 . By Corollary 3.20, $2m'_0 + 2m_0 \in W_1$. Since $2m'_0 < 2m'_0 + 2m_0 < 2m_0$, $m'_0 = -m_0$. This contradicts to 5) in Corollary 3.20. Therefore, for all $2m \in W_1$, $m \geq m_0$.

If there is $m > n_0$ such that $2m+1 \notin U_1$, then $f(2m+1) = 0$. Let $2m' \in U_2$ be the least one which satisfies $m' > n_0$. From $f(2m' + 2n_0 + 1) = 0$ and $n_0 < 0$, we get $2m' + 2n_0 < 2m'$. Therefore, $2m' + 2n_0 < 2n_0$, and $m' < 0$. By the nature of n_0 , we get the contradiction $n_0 > m'$. Therefore, for all $2n+1 \in U_1$, $n > n_0$.

Summarizing above discussion, we have that for all $m, n \in \mathbb{Z}$, $m < m_0$ and $n \leq n_0$, $f(2n+1) = 0$ and $f(2m) = 0$. Thanks to Corollary 3.20, $f(2n+1) = -1$ for $n > -n_0$ and $f(2n+1) \neq 0$ for $n_0 < n < 0$, $f(2m) = 0$ for all $0 < m < m_0$.

If there is $n \in \mathbb{Z}$ such that $0 < n < -n_0$ and $f(2n+1) = 0$. Let $n'' \in \mathbb{Z}$ be the least one satisfying $f(2n+1) = 0$, $0 < n < -n_0$. Then $f(2n_0 + 2n'' + 1) = 0$. We get the contradiction $2n_0 + 1 < 2n_0 + 2n'' + 1 < 2n'' + 1$. Therefore, for all $n \in \mathbb{Z}$, $0 < n < -n_0$, $f(2n+1) \neq 0$.

If there is $m \in \mathbb{Z}$ such that $-m_0 < m < 0$ and $f(2m) = 0$. Let $m'' \in \mathbb{Z}$, $-m_0 < m'' < 0$ be the largest one satisfying $f(2m'') \neq 0$. Then $f(2m_0 + 2m'') \neq 0$. But $2m'' < 2m_0 + 2 < m'' < 2m_0$. We get the contradiction. Therefore, there exist $m_0, l_0 \in \mathbb{Z}$, $m_0 > 0$ and $n_0 < 0$, such that

$$W_1 = \{2m | m \in \mathbb{Z}, m \geq m_0\}, W_2 = \{2m | m \in \mathbb{Z}, m < m_0\}, \\ U_1 = \{2n+1 | n \in \mathbb{Z}, n > n_0\}, U_2 = \{2n+1 | n \in \mathbb{Z}, n \leq n_0\}.$$

Similar discussion, if there is $m \in \mathbb{Z}$, $m < 0$ such that $f(2m) \neq 0$, then there exist $m_0, l_0 \in \mathbb{Z}$, $m_0 < 0$ and $n_0 > 0$, such that

$$W_1 = \{2m | m \in \mathbb{Z}, m \leq m_0\}, W_2 = \{2m | m \in \mathbb{Z}, m > m_0\}, \\ U_1 = \{2n+1 | n \in \mathbb{Z}, n < n_0\}, U_2 = \{2n+1 | n \in \mathbb{Z}, n \geq n_0\}.$$

(ii). Now we discuss the characteristic of the map f .

From above discussion, we need to discuss the case that $f(2m) \neq 0$ if and only if $m \geq m_0 > 0$, and $f(2n+1) \neq 0$ if and only if $n > n_0$, $n_0 < 0$.

From Corollary 3.20, Eq (8) and Eq (9), for all positive $l, k, s \in \mathbb{Z}$, $l \neq k$,

$$(21) \quad (f(2m_0 + 2s) + 1)(f(2n_0 + 2k + 1) + 1)(f(2n_0 + 2l + 1) + 1) = 0,$$

$$(22) \quad (f(2n_0 + 2s + 1) + 1)(f(2m_0 + 2k) + 1)(f(2m_0 + 2l) + 1) = 0.$$

Then we have

- the case $f(2m) = -1$ for all $m \in \mathbb{Z}, m \geq m_0$.

If $f(2n+1) = -1$, for all $n > n_0$, we obtain case 1).

If there is $n_1 \in \mathbb{Z}, n_1 > n_0$ and $f(2n_1+1) \neq -1$. By Corollary 3.20, Eq (8) and Eq (9), we have $l_0 \geq -3$, and

$$f(2n+1) \begin{cases} = -1, & \text{if } n \geq -n_0, \\ \neq 0, & \text{if } n_0 < n < 0, \\ = -1, & \text{if } 0 < n < -n_0; \end{cases} \quad \text{or} \quad f(2n+1) \begin{cases} = -1, & \text{if } n_0 < n < 0, \\ \neq 0, & \text{if } 0 < n < -n_0. \end{cases}$$

Therefore, if $l_0 = -3$, we get 2) and 3). If $l_0 = -2$, we obtain case 4).

•• If there is unique $m_1 \in \mathbb{Z}, m_1 \geq m_0$ such that $f(2m_1) \neq 0, -1$. Then by Eq (21), $f(2n+1) = -1$ for all $n \in \mathbb{Z}, n > n_0$; or there is unique $n_1 \in \mathbb{Z}, n_1 > n_0$ such that $f(2n_1+1) \neq 0, -1$, and $f(2n+1) = -1$ for all $n \in \mathbb{Z}, n > n_0$ and $n \neq n_1$. We obtain $f(2n+1) = -1$ for $n > n_0$. This is case 5). If there is $n_1 > n_0$ such that $f(2n_1+1) \neq -1$, we obtain case 6).

••• If the subset $S = \{m_k | m_k \in \mathbb{Z}, m_k \geq m_0, f(2m_k) \neq 0, -1, k \in \mathbb{Z}\}$ is non-empty. By Eq (21) and Eq (22), $S = \{m_1\}$, or $S = \{m_1, m_2\}$. we obtain 5) and 6), and 7).

The case (2) ($m_0 < 0$ and $n_0 > 0$) follows from the similar discussion. □

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